**Theory**

We want to calculate thermodynamic expectation values

We need to diagonalize the Hamiltonian. For this to work, we need to find its matrix elements. The Hamiltonian is

Several remarks are in order. Firstly, this Hamiltonian conserves the total spin, i.e. when applied to any state it does not change the number of spins up (or down) in this state. This means that in the basis it is block-diagonal. We can split up the state-sum as follows:

denotes the subspaces, denotes the basis states within this subspace. We find that

Hence:

The subspaces can be labelled by the number of spins up. There are such subspaces; write

Then,

Also, the Hamiltonian is symmetric under a sign flip. More precisely, we have

Without the magnetic field, then, we know that

i.e. it does not matter whether we look at the state as having spins up or spins down. This then further simplifies the equation

We need to go from to spins up, that’s the first half of the subspaces that are equivalent to the remaining .

This is because for even, there is an odd () number of such subspaces. We count all with up until spins up. Then there is the one with spins up and down, and the remaining ones are equivalent to the previous .

Now, switch on the magnetic field, we get

In subspace , the last term is always equal to

Hence,

We learned that

Now, exchange , then

Plugging this into the formula above, we get

where the last term corresponds to the case where is even. In this case, the middle subspace has .

**The Hamiltonian**

We encode all spin combinations as binary numbers from to . In the for-loop, we get the state .

The -term and the -term are diagonal: for each diagonal entry you iterate from to and add

for the -term and

for the -term. If we are within a subspace, we can simply add

to every diagonal entry.

We want the expectation value of several operators, including magnetization:

For the -term, we will have to be more careful. We iterate from to and look at the bit-representation with digits of . This represents a spin state. Traversing the state from left to right, we notice that if two adjacent spins are unequal, they will contribute

to the spin-flip term; here, the other index is the number we get when we take and exchange the two bits.

We are interested in expectation values of and . But both are block-diagonal, just as , and furthermore, on each are just multiples of the unit matrix with a value

In the general formula

then, we know that . Once we have calculated the partition sums, it is very easy to calculate these expectation values.

**Results, Thermodynamics**

Magnetization per site, , (free paramagnet):

A graph with numbers and lines

Description automatically generated

A graph of a function

Description automatically generated

Magnetization per site, , :

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From here:

A graph with a line

Description automatically generated

A graph with a line

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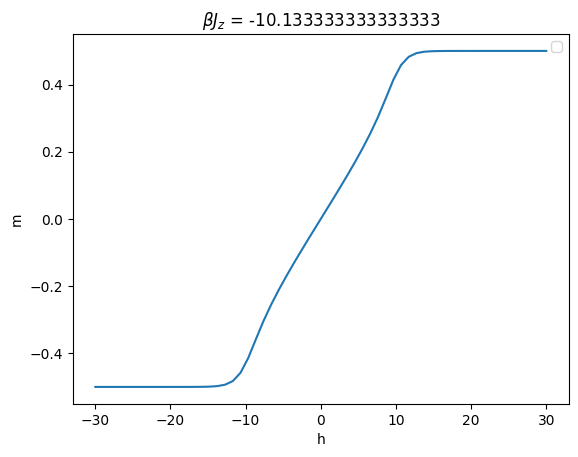
BEHAVIOUR: ANTI-FERROMAGNETIC

Same thing, but with negative coupling,

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Description automatically generated with medium confidenceA graph with a line

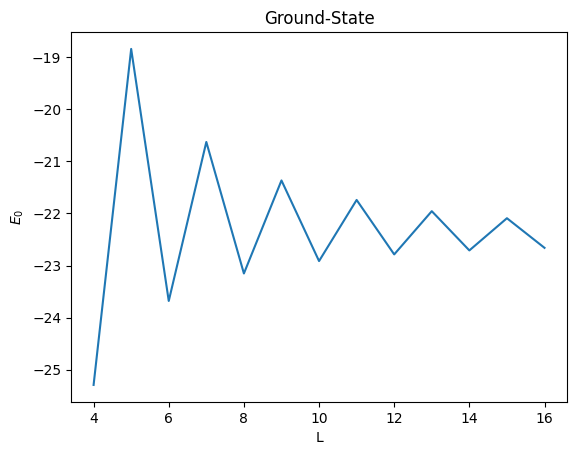
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Susceptibility as a function of

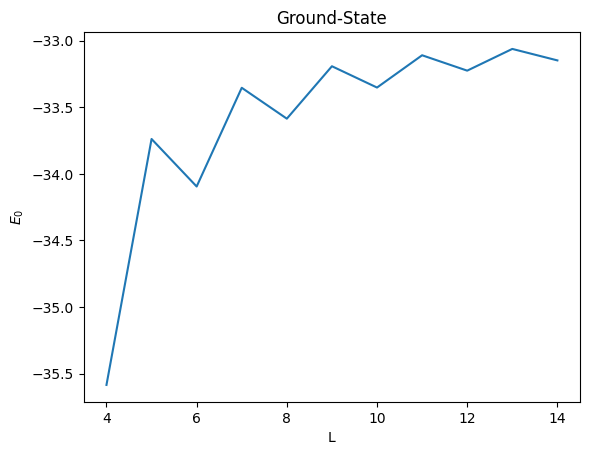
A graph of a function

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Ground-State energy: positive



Negative



**Results, Dynamics of the system**

Let’s also calculate the matrix exponential in non-diagonal form and find the time-evolution operator

A diagram of a graph

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A diagram of a graph

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